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MECHANICS OF EDGE EFFECTS ON FRICTIONLESS CONTACTS

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Abstract—This paper investigates edge effects on frictionless contacts. Edge phenomena are simulated by considering elastic contact near the edge of a wedge with varying wedge angle. Previous two-dimensional analysis is first reviewed and discussed to develop an understanding of the mechanics when edges are in contact. Three-dimensional wedge contact is then considered. The integral equation for rigid indentation of an incompressible wedge derived by the authors is presently modified to study indentation by a dissimilar elastic body. The analysis will focus on evaluating the effect of proximity to the wedge apex on contact stress, contact area geometry and contact compliance. The effect of varying wedge angles and elastic mismatch (encompassed by the Dundurs constant α) will also be quantified. The conditions under which elastic mismatch results in a singular contact stress when contact extends to the wedge apex, derived by Dundurs and Lee, will be discussed and used to interpret the present numerical results.

1. INTRODUCTION

Contact problems have been a topic of interest within the theory of elasticity for over 100 yrs. The classic paper by Hertz (1882) gave birth to an area of research which has grown into the present-day field of contact mechanics. Evidence that this topic has taken on its own identity is provided by the large amount of research over the last 100 yrs which has been dedicated to contact analysis. Classic books by Galin (1961), Gladwell (1980) and Johnson (1985) which review this research are further testaments to the emergence of contact mechanics as its own entity.

The foundation of contact mechanics analysis rests on the Hertz assumption. If the contact area is small compared to the geometry of the contacting bodies and if it is far removed from other surfaces, then the contacting bodies can be approximated as semi-infinite half planes or half spaces. This generally reduces the complexity of the original problem and simplifies the solution process, allowing closed form solutions in many cases. This Hertz assumption has provided a great stimulus for analyzing contact problems for half planes and half spaces, as displayed in the books noted above. Although a large percentage of contact mechanics problems occurring in practice may satisfy the Hertz assumptions, there are some that may not. Several paths lead away from the Hertz assumptions and it is one of these which is presently taken.

In some contact mechanics problems of practical interest, the contact area is not far removed from other external surfaces on the contacting bodies. These external surfaces, referred to as edges, may appear in close proximity to the contact region and thus violate one of the Hertz assumptions. Examples of this in engineering which motivated the authors' interest are rail/wheel contact and contacts involving cutting tools. If this particular aspect of the Hertz assumptions is violated, the question becomes one of how inaccurate are Hertzian stress fields when applied to these problems. The answer to this question requires the careful examination of concentrated contacts near edges. From an elasticity point of view, the study of edge effects utilizes an infinite wedge geometry. One face of the wedge represents the contact surface of interest while the other face represents the additional surface (which may or may not be stress free) near the contact region. The apex of the wedge represents the edge near the contact region. The study of edge effects on contacts thus requires the analysis of contact problems for a wedge geometry. Contact problems for a wedge geometry have received some attention in two-dimensional analysis. Until recently, wedge contact in three-dimensions was largely unexplored. As an aid in understanding the subsequent three-dimensional analysis, the following section reviews the research on edge effects in two-dimensional contact which reveals some interesting and somewhat unexpected features of this problem.

2. EDGE EFFECTS IN TWO-DIMENSIONAL CONTACT ANALYSIS

The application of elasticity theory to stress analysis for a two-dimensional wedge geometry is well developed. Books such as Sneddon (1951) and Uflyand (1965) review the application of the Mellin transform to this geometry. These methods have been used by many researchers, too numerous to name, for performing stress analysis of a wedge shaped geometry. The number dwindles significantly, however, when one considers the analysis of mixed boundary value problems of contact type for wedges, as opposed to the case where only stress boundary conditions are applied to the wedge surfaces.

The first such investigation of a contact problem for a wedge might be the analysis of Matczynski (1963). He considered a condition on the normal displacement over a region on one face extending to the wedge tip. Additional results in the Russian literature (which have been translated) are provided by Lutchenko (1966), Aleksandrov and Kopasenko (1968), Lutchenko and Popov (1970), Nuller (1972) and Lutchenko (1972). These papers all have two things in common. First, all consider frictionless indentation. Secondly, they all formulate the integral equation necessary but give very little or no numerical results for the contact problem. In fact, only Lutchenko (1966) gives an approximate numerical result for indentation by a flat punch but it is unclear from his paper how far the punch is from the edge.

The English literature also contains results on this topic. Srivastav and Narain (1965) may have been the first to formulate the dual integral equations necessary for symmetric frictionless indentation. They reduced them to a Fredholm integral equation of the second kind but gave no numerical results. The first numerical investigation for wedge contact was conducted by Gerber (1968). Using Hetenyi's reflection method, Gerber constructed a piecewise constant approximation for frictionless contact stress analysis of a quarter plane. He considered rigid indentation and allowed the contact area to extend to the corner of the quarter plane. His conclusions when contact extends to the corner are subsequently discussed. Erdogan and Gupta (1976) used Mellin transforms to conduct a detailed investigation of frictionless contact on a two-dimensional wedge of arbitrary angle. They studied a flat indenter under a central or non-central force and allowed the indenter to approach the wedge apex. Their results provided the contact stress and the coefficients of the square root singularities at the edges of contact when contact did not extend to the edge. When contact reached the wedge apex, they evaluated the singularity of the contact stress there as a function of wedge angle. An interesting conclusion for small wedge angles was the loss in contact that may occur as the wedge apex was approached, which will be discussed further below. Erdogan and Arin (1976a,b) also provided a detailed numerical investigation of wedge contact problems with a general wedge angle. Erdogan and Arin (1976a) analyzed the wedge splitting problem which amounted to frictionless contact symmetrically distributed on both wedge faces. Both sharp cornered and smooth punch profiles were considered. The analogous problem including Coulomb friction was treated by Erdogan and Arin (1976b). Erdogan and Civelek (1979) considered the additional effect of a crack along the mid-plane of the wedge. The numerical results in these papers reveal the effect of proximity to the edge on the contact stress parameters. For a flat indenter, numerical results displayed how the coefficients of the square root singularities at the edges of contact changed as the apex was approached. It was also concluded that the solution for a circular indenter did not differ significantly from the half plane results, although the circular punch was not

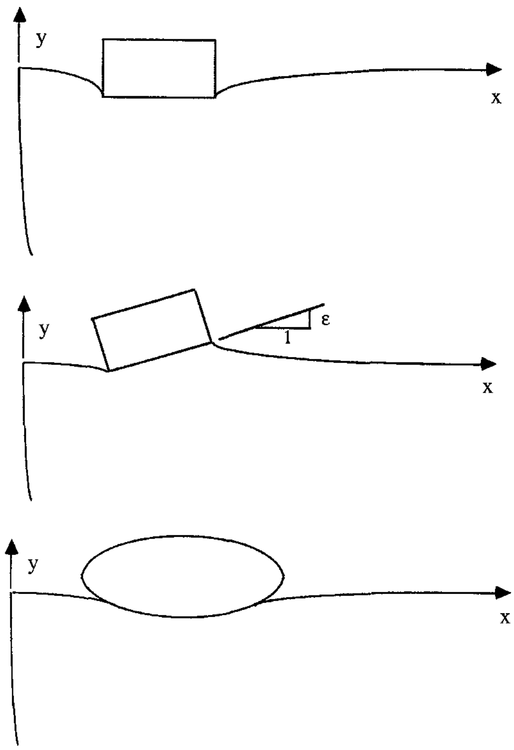


Fig. 1. Frictionless indentation of a quarter plane. (a) Flat indentation of a quarter plane. (b) Inclined indentation of a quarter plane. (c) Parabolic indentation of a quarter plane.

allowed to closely approach the apex. The closest indenter position for the flat indenter was at $d/c = 4/3$, where d was the distance from the wedge apex to the center of contact and c was the half contact length. The circular punch was positioned much farther away.

Although the above analysis provides a great deal of insight for contact near the edge, the case of contact at the edge needs further consideration. It may be that Gerber (1968), Erdogan and Gupta (1976) and Hanson and Keer (1989) are the only investigations of contact at the edge. Gerber (1968) and Hanson and Keer (1989) studied frictionless indentation of a quarter plane while the general wedge was considered by Erdogan and Gupta (1976). To illustrate the effect of contact at the edge, consider Fig. 1(a)–(c) which displays rigid indentation of a quarter plane. The integral equation for this problem was derived by Hanson and Keer (1989) in the form

$$\frac{4\mu}{\kappa+1}\frac{\mathrm{d}v\,(x)}{\mathrm{d}x} = \frac{1}{\pi}\int_a^b \frac{\sigma(\xi)}{\xi-x}\mathrm{d}\xi - \frac{1}{\pi}\int_a^b \frac{\sigma(\xi)}{\xi+x}\mathrm{d}\xi + \int_a^b \sigma(\xi)K(x,\xi)\,\mathrm{d}\xi, \quad a < x < b, \tag{1}$$

where $v\,(x)$ is the displacement on the upper surface in the y direction, $\sigma(\xi)$ is the contact pressure and $\mathrm{K}(x, \xi)$ is a generalized Cauchy kernel. Here μ, v are the quarter plane shear modulus and Poisson ratio with $\kappa = 3\text{–}4v$ for plane strain and $\kappa = (3-v)/(1+v)$ for plane stress. Considering rigid indentation out to the corner ($a = 0$), it was shown that a bounded displacement gradient at the corner leads to a bounded contact pressure there. If $\sigma(0) = D$, then Hanson and Keer (1989) showed that

$$\operatorname*{Lim}_{x \to 0^+}\left\{\frac{4\mu}{\kappa+1}\frac{\mathrm{d}v\,(x)}{\mathrm{d}x}\right\} = -\frac{\pi D}{4} + O(x). \tag{2}$$

This result is interesting since intuition might lead one to speculate that a corner in contact would cause a singular stress there.

It is important to note the following. The above result implies that if contact does not extend to the edge, the corner remains at a right angle since the shear strain is zero and undergoes no rotation but translates like a rigid body (since D above would be zero in this case). Considering a rigid, flat and upright indenter [Fig. 1(a)], the contact stress is in general square root singular at the edges of contact. As the indenter approaches the edge, the coefficient of the square root singularity diminishes on the side closest to the edge. When the edge is reached, the square root singularity and in fact the contact stress itself will vanish at the corner (Erdogan and Gupta, 1976; Hanson and Keer, 1989). This results since the flat indenter has a zero displacement gradient and thus D in the above equation must also be zero. This is also evident from Fig. 4.2 in Gerber (1968) although his conclusion is that the contact stress at the corner is bounded, not zero. At the indenter corner away from the edge, the coefficient of the square root singularity increases as the edge is approached (see Table 1 in Erdogan and Gupta, 1976; Fig. 2 in Hanson and Keer, 1989). Finally, for wedge angles greater than a quarter plane, Erdogan and Gupta (1976) show that when contact of a flat punch extends to the apex, the contact stress is singular there with the power of the singularity increasing with larger wedge angles.

Now consider the rigid, flat and inclined punch [Fig. 1(b)] loaded by a moment and a zero net force. In this case the punch will rotate an amount ε. Again the stress is square root singular at the edges of contact. As the indenter approaches the quarter plane corner, the strength of this singularity diminishes on both sides of contact (Fig. 3 in Hanson and Keer, 1989). When contact extends to the edge, the singularity vanishes there and the stress at the corner is bounded. The value of this stress comes from eqn (2) with $\mathrm{d}v\,(x)/\mathrm{d}x = \varepsilon$ as

$$\sigma(0) = -\frac{16\mu}{\pi(\kappa+1)}\varepsilon. \tag{3}$$

This case was also considered by Gerber (1968); however, he concluded that when contact extended to the corner, the contact stress was infinite there, as the numerical results in his Fig. 4.3 indicated.

The above equation has an interesting implication for upright cylindrical indentation as shown in Fig. 1(c). In this case, if the indenter center is to the right of the corner the displacement gradient above the corner is negative. Equation (3) then requires a tensile stress. Since this is physically impossible, it may be concluded that when the indenter center is to the right of the corner, contact will not extend to the corner for any magnitude of the contact force. As the force is increased, contact will expand out on both sides but the increase will be larger on the side away from the edge. On the edge side of contact, the contact length will increase at a diminishing rate as the force is increased but never reaches the corner. When the indenter center is directly over the edge, the contact stress there is zero since the displacement gradient is also zero, which is in contrast to the numerical result and conclusion by Gerber (1968) that the contact stress was bounded but non-zero.

The numerical results discussed above indicate a general property of frictionless contact near a vertical edge. The presence of the edge causes a loss in stiffness as compared to a half plane. For the flat upright indenter, the stiffness loss causes the strength of the square root singularity at the edge of contact away from the corner to increase as the corner is approached (in contrast to the decrease at the edge closest to the corner). For the tilted indenter loaded only by a moment, the strength of the singularity at both edges decreases as the corner is approached. The cylindrical indenter has perhaps the most intriguing behavior. As the indenter approaches the corner with the contact length held fixed, the stress retains almost a Hertz distribution. This is shown in Fig. 2(a) (taken from Hanson and Keer, 1989) where c is the half contact length, l is the distance from the quarter plane corner to the center of contact and $\bar{x}$ is a non-dimensionalized distance in the contact region. This behavior is also illustrated by Fig. 4.6 in Gerber (1968). In fact, when the indenter tip is at the corner, $l/c = 1$ and the contact stress profile differs only slightly from the Hertz result. However, the contact is not Hertzian since there is a shift in the center of the contact area off from the center of the indenter, in a direction away from the edge. This

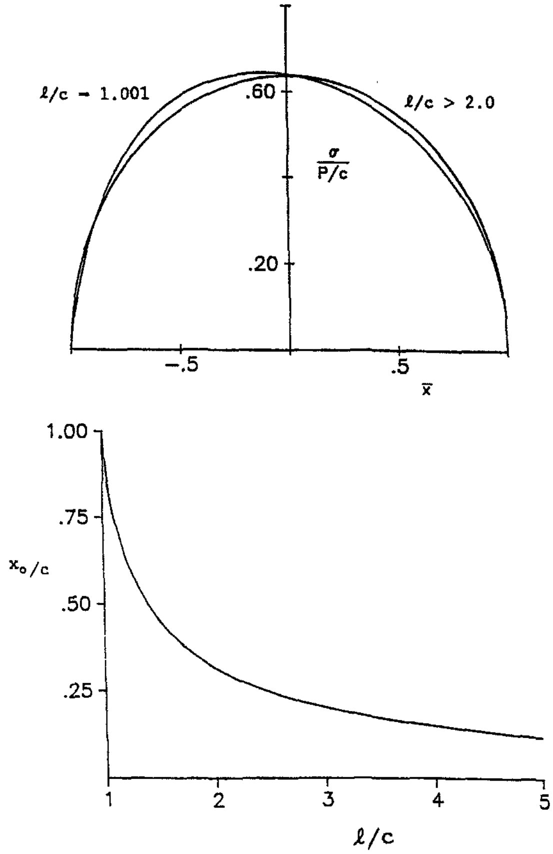


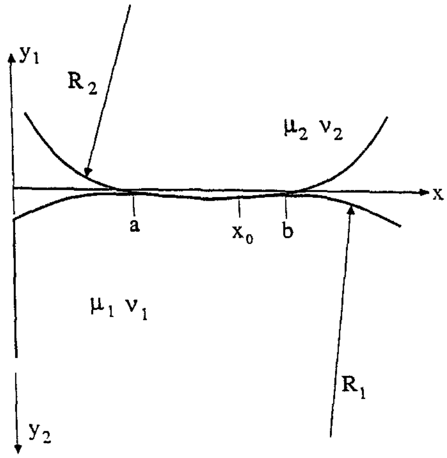
Fig. 2. Cylindrical indentation of a quarter plane. (a) Contact stress for cylindrical indentation of a quarter plane. (b) Shift of the center of contact off from the center of the indenter.

is illustrated in Fig. 2(b), also taken from the paper by Hanson and Keer, where x_o is the shift in the center of contact off from the indenter tip. The contact stress is always maximum near the center of contact but this point is not the center of the indenter, where maximum penetration occurs.

Now consider frictionless contact of dissimilar elastic bodies. Figure 3(a) illustrates this contact when one of the bodies has a vertical traction free surface near the contact region (a quarter plane) and the upper body is approximated as a half plane. The integral equation was derived by Hanson and Keer (1989)

$$\frac{4\mu_1}{\kappa_1+1}\left\{\frac{dv_1(x)}{dx}+\frac{dv_2(x)}{dx}\right\}=\frac{2}{(1+\alpha)\pi}\int_a^b\frac{\sigma(\xi)}{\xi-x}\,d\xi-\frac{1}{\pi}\int_a^b\frac{\sigma(\xi)}{\xi+x}\,d\xi$$
$$+\int_a^b\sigma(\xi)K(x,\xi)\,d\xi,\quad a<x<b,$$
$$\alpha=\frac{(\kappa_1+1)\Gamma-(\kappa_2+1)}{(\kappa_1+1)\Gamma+(\kappa_2+1)},\quad \Gamma=\frac{\mu_2}{\mu_1},\tag{4}$$

where α is a Dundurs' constant. When the upper body becomes rigid, $\Gamma\to\infty$, $\alpha\to 1$, $dv_2(x)/dx\to 0$ and eqn (1) is recovered. Note that the limits on Γ and α are: $0\leqslant\Gamma\leqslant\infty$, $-1\leqslant\alpha\leqslant 1$. Consider $a>0$ so contact does not extend to the edge. When $\alpha=-1$, the half plane term dominates (the quarter plane is rigid) and the edge has no effect. The edge effect is maximum when the coefficient of the half plane term is minimized, which occurs when $\alpha=1$. Therefore, a rigid indenter on a quarter plane provides the largest deviation from classical half plane results.



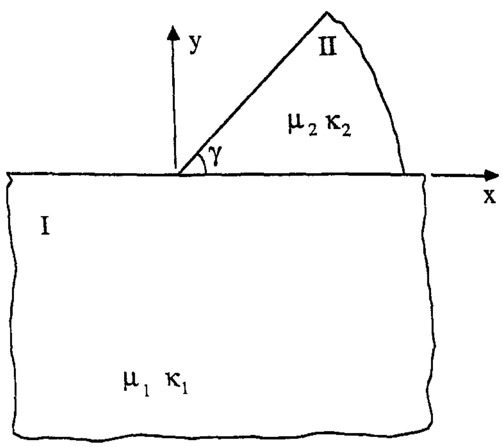


Fig. 3. Frictionless contact between a wedge and a dissimilar elastic half plane. (a) Frictionless contact of dissimilar elastic bodies near a vertical free surface (from Hanson and Keer, 1989). (b) Frictionless contact between a wedge and a dissimilar elastic half plane (from Dundurs and Lee, 1972).

For contact extending to the corner ($a = 0$) it was shown above that the contact stress is bounded for a rigid indenter with a bounded displacement gradient. This is not the case for general dissimilar elastic contact. It was shown by Dundurs and Lee (1972) that a singularity may exist in the contact stress at the corner. Figure 3(b) illustrates the problem they considered. An elastic wedge of total angle γ is pressed into a dissimilar elastic half plane. Using a Mellin transform analysis they showed that the stress $\sigma(x)$ in the contact region for $x \to 0^+$ may exhibit one of the three behaviors: $O(x^{p-1})$, $O(\ln x)$ or $O(1)$ depending on p which is a solution to the equation

$$[1+\alpha^D]\cos(p\pi)[p^2\sin^2\gamma-\sin^2 p\gamma]-\tfrac{1}{2}[1-\alpha^D]\sin(p\pi)[p\sin(2\gamma)+\sin(2p\gamma)]=0. \quad (5)$$

Note that α^D is as defined above but the subscripts on the elastic constants for the half plane and the wedge are reversed in Fig. 3(b) as compared to Fig. 3(a) (hence the superscript D is used to differentiate between the two). Thus α^D refers to the material constants as defined in Fig. 3(b) whereas α refers to Fig. 3(a). From a numerical study they concluded that p was always real and varied between one-half and one for $0 < \gamma < \pi$. Thus the contact stress at the edge ranged from being bounded to a power law singularity no stronger than square root singular. They determined that the $\ln x$ singularity occurs at the transition region between the power law singular behavior and the bounded behavior.

A singularity analysis was also conducted by Hanson and Keer (1989) on the integral eqn (4) pertinent to the problem shown in Fig. 3(a). Assuming a solution to eqn (4) of the form $\sigma(\xi) \to \xi^{-s}$ for $\xi \to 0$, they found that s must satisfy the equation

$$[1-\alpha]\cos(s\pi)\left[(1-s)^2-\cos^2\left(\frac{s\pi}{2}\right)\right]+\tfrac{1}{2}[1+\alpha]\sin^2(s\pi) = 0. \tag{6}$$

If it is noted that $\alpha^D = -\alpha$ and $p = 1-s$, eqns (5) and (6) are identical for $\gamma = \pi/2$. One may thus conclude the following for frictionless contact of dissimilar elastic bodies, where one of the bodies has an inclined free surface and contact extends to the corner. If the total wedge angle is greater than 90°, there will be a power law singularity in the contact stress at the corner for all combinations of elastic constants. For a 90° wedge angle, there will be a singularity at the corner if the half plane is not rigid and a rigid half plane will give a bounded stress at the corner. If the wedge angle is less than 90°, the contact stress at the edge may be either bounded or singular, depending on the value of α for the contacting bodies. These conclusions are based on the numerical analysis revealed in Fig. 2 in Dundurs and Lee (1972) and the above considerations.

Although the above comments are directed at a wedge angle of less than π, an additional conclusion can also be made. For the special case of a total wedge angle greater than π, rigid flat indentation at the apex will lead to a singular contact pressure there as shown by Erdogan and Gupta (1976). In this special case, as the wedge angle increases from π to 2π, the power of the contact stress singularity increases from 0.5 to 1.0.

3. EDGE EFFECTS IN THREE-DIMENSIONAL CONTACT ANALYSIS

Stress analysis for a three-dimensional wedge has received little attention in the past literature. To set terminology, a three-dimensional wedge is geometrically the two-dimensional wedge above but the loading varies in a direction parallel to the wedge apex and is thus a three-dimensional problem. Few analytical solutions exist for this geometry except in the special case of a 360° total wedge angle (which is equivalent to an infinite body with a half plane crack). The Papkovich–Neuber potential function method developed for a general wedge angle is reviewed by Uflyand (1965). The first analytical point force solution might have been found by Efimov and Efimov (1986) who solved for point normal loading on one face of an incompressible wedge and at the wedge tip. This solution was derived independently by Hanson and Keer (1991) while the extension to an internal point force and point tangential loading on the surface was made by Hanson *et al.* (1994). In all of these previous analyses only the potential functions have been found. The internal elastic field for concentrated normal force and point moment loading on one face or at the wedge tip has only recently been evaluated by Hanson (1995).

The only other stress analysis which has been conducted for this geometry is when the total wedge angle is 90°, termed an elastic quarter space. Hetenyi (1970) was the first to consider this problem when he investigated point normal loading of a quarter space. He used a reflection iterative scheme of overlapped symmetrically loaded half spaces which was an extension of his previous elastic quarter plane study (Hetenyi, 1960). This quarter space problem was reformulated by Keer *et al.* (1983). They used only two overlapped half spaces with unknown symmetric pressure distributions to formulate a coupled pair of integral equations which were numerically solved in the Fourier transform domain. This approach was an extension of a direct formulation developed by Sneddon (1971) for the quarter plane problem. Keer *et al.* (1983) considered both point normal and tangential loading. A finite element analysis was conducted by Bower *et al.* (1987) to estimate a ratchetting limit for plastic deformation in rails. The coupled integral equations for the quarter space were re-solved directly (not in the Fourier transform domain) by Hanson and Keer (1990a). Since this solution did not require the Fourier transform of the applied loading, any loading could be considered. They studied ellipsoidal contact on one or simultaneously on both faces of the quarter space. It is important to note that the difficulty

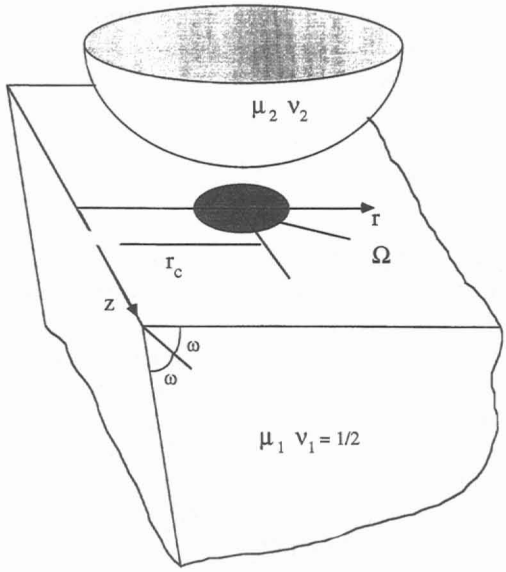


Fig. 4. Frictionless contact of a dissimilar elastic body with a three-dimensional incompressible wedge.

in solving the quarter space problem by this integral equation method is the logarithmic singularity in the coupled equations. This issue was first revealed in the quarter space analysis of Keer *et al.* (1983) and is also discussed in detail by Hanson and Keer (1989, 1990a,b).

Until recently, the contact problem for a three-dimensional wedge was largely unexplored. The first investigation of frictionless contact was the quarter space study by Gerber (1968). He looked at a flat rectangular punch and gave results for the stress distribution. The papers by Babeshko and Berkovich (1972) and Berkovich (1974) discuss certain contact problems but no useful results are given. Keer *et al.* (1984) considered rigid cylindrical indentation of a quarter space using a patch solution and their previous method. Hanson and Keer (1991) used their point force solution to formulate the integral equation for frictionless rigid contact of an incompressible wedge and studied spherical indentation of an incompressible quarter space. This numerical analysis was extended to other wedge angles and to simultaneous contact on both faces by Hanson and Keer (1993). The results of this three-dimensional contact analysis will be discussed below in the light of the insight provided by the two-dimensional behavior. Before this, however, consideration is given to dissimilar elastic contact for a three-dimensional wedge.

The problem under consideration is shown in Fig. 4, where both bodies are considered as isotropic. For the upper body, the Hertz assumptions will be applied and it will be approximated as an elastic half space. The half space has a shear modulus μ_2 and Poisson ratio ν_2 while μ_1 is the shear modulus for the wedge with $\nu_1 = 1/2$. To derive the integral equation for this contact problem, the solutions for the normal surface displacement caused by point force loading on an isotropic half space and a three-dimensional wedge are needed. The half space results can be easily found in Johnson (1985) while Hanson and Keer (1991) evaluated the needed results for a three-dimensional incompressible wedge. Proceeding in the conventional manner leads to the integral equation

$$\delta - f(r, z) = \left\{\frac{1-\nu_2}{2\pi\mu_2} + \frac{1}{4\pi\mu_1}\right\} \int\!\!\int \frac{p(r_o, z_o)\, \mathrm{d}r_o\, \mathrm{d}z_o}{[(r-r_o)^2 + (z-z_o)^2]^{1/2}}$$
$$- \frac{1}{4\pi\mu_1} \int\!\!\int p(r_o, z_o) \left\{\frac{1}{(rr_o)^{1/2}} \int_o^\infty \frac{F(\tau, \omega)}{G(\tau, \omega)} P_{i\tau - 1/2}(\cosh(\beta))\, \mathrm{d}\tau\right\} \mathrm{d}r_o\, \mathrm{d}z_o, \quad (7)$$

where the double integral is taken over the contact region Ω. Here $p(r_o, z_o)$ is the unknown contact pressure, δ is the relative approach, $f(r,z)$ is the surface separation distance and $P_{i\tau-1/2}(\cosh(\beta))$ is a Legendre function of the first kind with $\cosh\beta = [r^2+r_o^2+(z-z_o)^2]/(2rr_o)$. The functions $F(\tau,\omega)$ and $G(\tau,\omega)$, where 2ω is the total wedge angle, are defined by eqns (36) and (37) in Hanson and Keer (1991).

Approximating the upper body with a spherical curvature of radius R and the wedge with planar faces, the integral equation is recast as

$$\delta-\frac{1}{2R}[(r-r_c)^2+z^2] = \left\{\frac{1}{2\pi(1+\alpha)}\right\}\int\int\frac{\bar{p}(r_o,z_o)\,\mathrm{d}r_o\,\mathrm{d}z_o}{[(r-r_o)^2+(z-z_o)^2]^{1/2}}$$

$$-\frac{1}{4\pi}\int\int\bar{p}(r_o,z_o)\left\{\frac{1}{(rr_o)^{1/2}}\int_o^\infty\frac{F(\tau,\omega)}{G(\tau,\omega)}P_{i\tau-1/2}(\cosh(\beta))\,\mathrm{d}\tau\right\}\mathrm{d}r_o\,\mathrm{d}z_o, \quad (8)$$

where $\bar{p}(r_o,z_o) = p(r_o,z_o)/\mu_1$ is the non-dimensionalized contact pressure, r_c is the point of initial contact, $\kappa = 3$–4ν for three-dimensional problems and the Dundurs' parameter α is given in eqn (4). Note that the first term on the right-hand side above is the result for two half spaces (the lower one being incompressible) while the second term results from the effect of the wedge.

Consider the case when contact does not extend to the edge. As α approaches its lower limit of $\alpha = -1$ ($\Gamma = 0$), the half space term dominates and the contact becomes Hertzian. This corresponds to a spherical elastic body being pressed into a rigid plane. At its maximum value of $\alpha = 1$ ($\Gamma = \infty$), the half space term gives its minimum contribution and the maximum deviation from Hertzian contact caused by the wedge is reached. This case corresponds to rigid indentation of a wedge and the integral equation derived by Hanson and Keer (1991) is recovered. It may then be concluded that the largest deviation from Hertzian contact occurs for rigid indentation of the wedge. As the shear modulus of the wedge increases relative to the half space, the contact approaches the Hertz result. This is consistent with the two-dimensional quarter plane results above.

If contact extends to the edge, a different situation occurs. In planes perpendicular to the edge near the center of contact, plane strain conditions will be approximated. The two-dimensional results above then predict that a power law singularity will be present in the contact stress at the edge. Under the present three-dimensional conditions, the coefficient of this singularity will vary along the edge. In the special case of a 90° wedge (quarter space) and a rigid upper body, the contact stress along the edge should be bounded as predicted by the quarter plane study. The above conclusions are confirmed by the following numerical results.

The integral equation (8) is solved numerically by assuming a piecewise constant approximation to $\bar{p}(r_o,z_o)$. The details of the integrations are described in Hanson and Keer (1991). If the relative approach δ and the radius R are specified, the contact region Ω is *a priori* unknown (it may not be circular due to the wedge effect). Thus an oversized blanket of elements is used to estimate the contact area and an iteration procedure is applied to eliminate patches with a tensile contact pressure. The indenter radius is taken as $R = 8/\pi$ in the numerical study and δ is given the values $\pi/8$, $\pi/16$, $\pi/32$ and $\pi/64$. The Poisson ratio of the upper body is $\nu_2 = 0.3$ while stiffness ratios of $\Gamma = \frac{1}{3}$, 1 and ∞ are used, where $\Gamma = 0$ is Hertzian contact if contact does not extend to the edge. For comparative purposes, the Hertzian radius a and the maximum contact pressure $\bar{p}_{\max}$ are

$$a = (\delta R)^{1/2}, \quad \bar{p}_{\max} = \frac{8\Gamma}{\pi[\Gamma+2(1-\nu_2)]}\left(\frac{\delta}{R}\right)^{1/2}. \quad (9)$$

The value of R and $\delta = \pi/8$ were chosen by Hanson and Keer (1991) so that the Hertz results become $a = 1$ and $\bar{p}_{\max} = 1$ when $\Gamma = \infty$. The distance r_c is taken as ∞, 3.0, 1.5, 1.0, 0.5 and 0.0 where $r_c = \infty$ implies the half space result while $r_c = 0$ places the tip of the

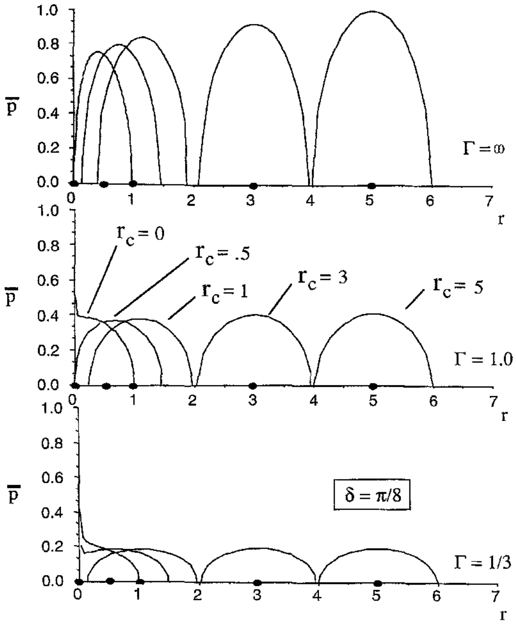


Fig. 5. Contact stress through the center of contact for a 90° wedge (quarter space).

upper body at the wedge apex. Total wedge angles of $2\omega = 60, 90$ (quarter space) and 126° are used.

The first result is shown in Fig. 5 for contact with a quarter space when the relative approach is $\delta = \pi/8$. The contact stress through the center of contact perpendicular to the edge is plotted. The heavy black dots indicate the position of the indenter tip in each case. Considering the top figure for rigid indentation ($\Gamma = \infty$), if $r_c > 5$ the contact is Hertzian. As the indenter is brought closer to the wedge apex, the stress distribution remains essentially Hertzian but the peak stress, contact area and hence the contact force diminish. When $r_c = 0$, the indenter tip is on the edge and the contact stress at the edge goes to zero. This behavior is in exact agreement with the quarter plane results shown in Fig. 2(a). Using a different method, Keer *et al.* (1984) analyzed rigid indentation of a quarter space by a right circular cylinder lying on its side overhanging the edge for an arbitrary Poisson ratio. If the cylinder was not tilted, the indenter slope perpendicular to the edge was zero and their numerical results (Fig. 6 of their paper) also verified the contact stress tends to zero when Poisson's ratio was one-half. Their results for a compressible quarter space reveal that for a zero displacement gradient perpendicular to the edge the contact stress at the edge is bounded and non-zero. When the cylinder axis was tilted, a bounded stress for all values of Poisson's ratio was obtained although the presentation of the results does not allow one to verify if eqn (3) is satisfied for the incompressible case. Gerber (1968) considered a square flat indenter pressed into a quarter space and allowed contact to extend to the edge. He also concluded from his numerical results (Figs 5.4–5.6 of his thesis) that the contact stress was zero for incompressibility and non-zero bounded for other values of Poisson's ratio.

Dissimilar elastic contact is shown in the middle and bottom graphs of Fig. 5. Since as $\Gamma \to 0$ results tend towards the Hertzian case, there is less change as the edge is approached. However, once the edge comes into contact the stress becomes infinite there. For $\Gamma = 1.0$, this occurs when the indenter tip is pressed into the edge. When $\Gamma = 1/3$, this occurs also when $r_c = 0.5$. Since a piecewise constant approximation to the contact stress was used, no decisive conclusions can be drawn about the order or coefficient of the stress singularity. However, one can infer which conditions lead to a more severe state from the present results. For example, from the bottom graph in Fig. 5, when $r_c = 0$ the stress state is more severe at the edge than when $r_c = 0.5$. According to the two-dimensional analysis above, the order of the singularity is a function of only ω and α. Since these two parameters are

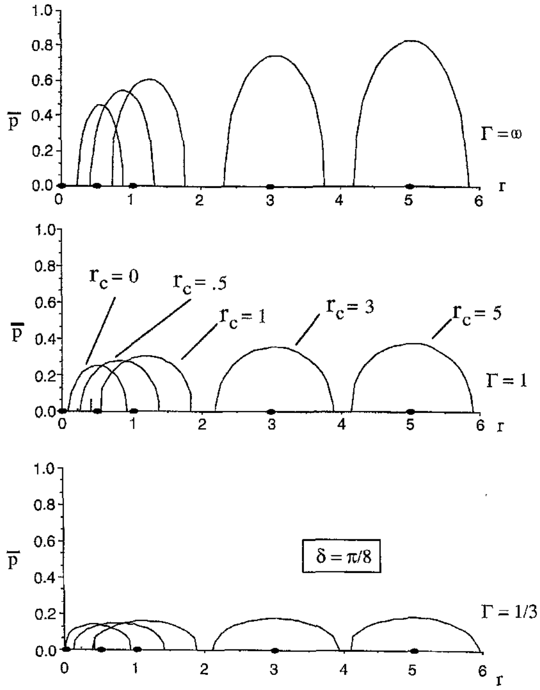


Fig. 6. Contact stress through the center of contact for a 60° wedge.

constant in the bottom graph, the magnitude of the normal displacement and possibly its gradient at the edge must have a significant impact on the coefficient of the singularity.

The following two figures display the wedge angle effect where again $\delta = \pi/8$. Figure 6 displays the stress through the center of contact for a 60° total wedge angle. In this case, the contact stress is still Hertzian in nature, but lower in magnitude than for the quarter space. This occurs since a reduction of the wedge angle causes the wedge to become more compliant, reducing the contact force. The shift in the center of contact from the wedge tip is very noticeable in this case. When $r_c = 0$, the indenter tip is pressed directly over the edge but the wedge apex pulls away and contact results over an area not extending to the edge. [This "pulling away" behavior for the 60° wedge was previously noted in the two-dimensional analysis of Erdogan and Gupta (1976) who considered rigid flat indentation.] Again this effect is most pronounced for the rigid indenter and diminishes as Γ is decreased. For the 60° wedge with this indentation depth, the edge never comes into contact and the stress is never singular. This is not the case for the 126° wedge shown in Fig. 7. Here the contact stress is singular for $r_c = 0.5$ and 0.0 for all values of Γ. Compared to the quarter space, the stress singularity at the edge is now much stronger. This is consistent with the results in Dundurs and Lee (1972), which predict the order of the singularity to increase with increasing wedge angle. As expected, the coefficient of the stress singularity is more severe for $r_c = 0$ as compared to $r_c = 0.5$.

Figure 8 reveals the alteration of the contact area geometry as the wedge apex is approached for a rigid indentation with $\delta = \pi/8$. Again the solid dot indicates the indenter tip position for each value of r_c. For the 126° wedge, the contact area is essentially circular or a truncated circle where it intersects the edge. As the wedge angle is reduced, there is a more pronounced shift of the indenter tip off from the center of the circle. The top graph reveals how the 60° wedge deforms away from the indenter. When $r_c = 0$, the indenter tip is directly over the edge but the contact area is semi-circular, not extending to the edge. The smaller wedge angle causes an increased compliance with almost a bending type phenomenon that deforms the wedge apex away from the contacting body.

The last results display the contact force P. In Fig. 9 it is shown how the contact force diminishes as the contacting body is moved closer to the edge with the indentation depth held fixed at $\delta = \pi/8$. Note that reducing the wedge angle gives an increase in compliance

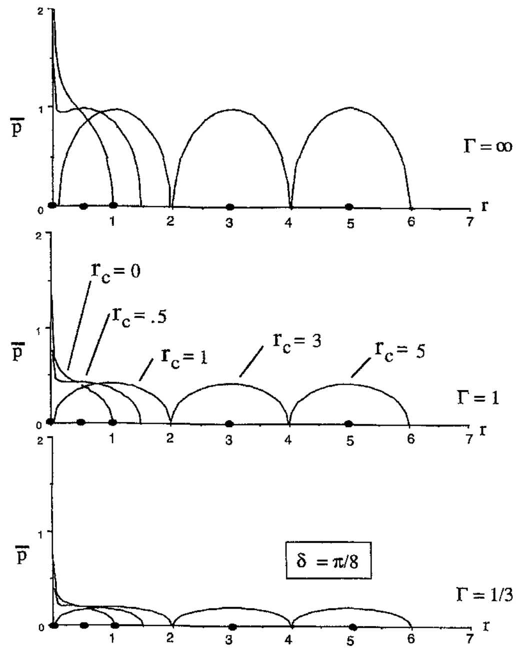


Fig. 7. Contact stress through the center of contact for a 126° wedge.

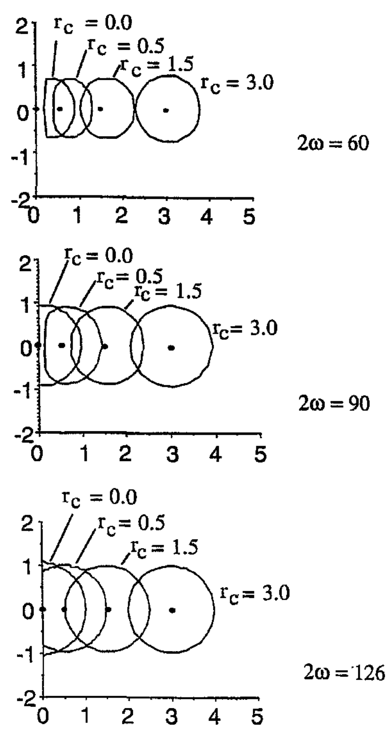


Fig. 8. Variation in contact area as the edge is approached for a constant indentation depth.

and a reduction in force. If all other factors are held constant, the contact force also decreases as Γ is reduced. As r_c increases, the force asymptotically approaches the Hertzian result. For smaller values of Γ, this approach is more rapid. Figure 10 illustrates the force deflection relation for a specific distance from the edge for rigid indentation of a quarter

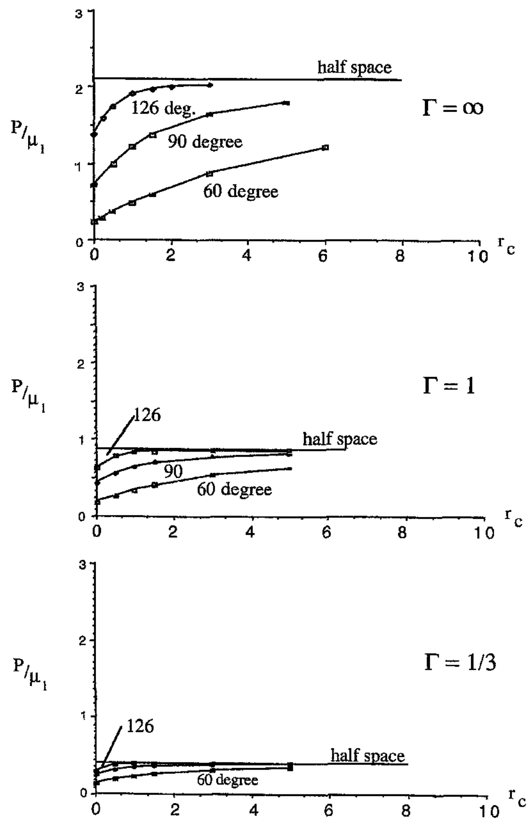


Fig. 9. Contact force variation as the edge is approached for a constant indentation depth.

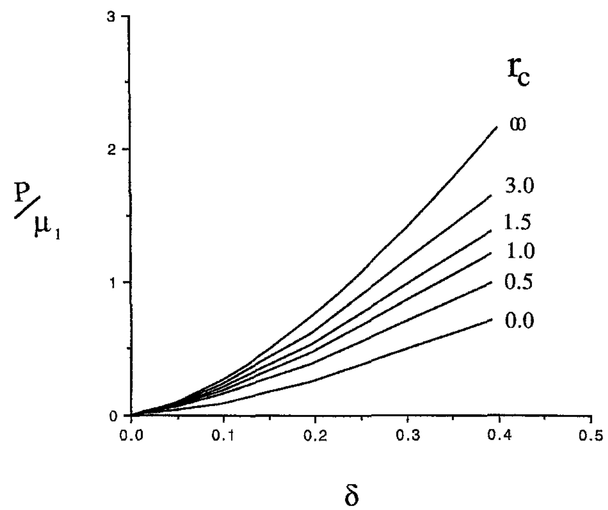


Fig. 10. Force deflection relation for a quarter space under rigid spherical indentation.

space. As the indenter tip is placed closer to the edge, there is a reduction in the stiffness of the contact.

4. CONCLUSIONS

Several interesting conclusions can be drawn concerning frictionless contact near an edge. From the two-dimensional results derived by Dundurs and Lee (1972), a stress singularity will in general occur for edges in contact. If the total wedge angle is between 90° and 180°, their results predict a stress singularity for all material combinations. For a 90° wedge, the stress will be bounded for a rigid indenter with magnitude given by eqn (3) while a non-rigid indenter will lead to a stress singularity. If the total wedge angle is less than 90°, the stress may be either bounded or singular, depending on the value of the Dundurs constant α for the contact. For rigid parabolic indentation of a quarter plane, the contact stress essentially retains a Hertzian distribution when contact extends to the edge. However, there is a shift in the center of contact from the indenter tip in a direction away from the edge.

These results were also confirmed in the three-dimensional incompressible wedge study. It may be concluded that Hertzian conditions prevail if $r_c > 5$ when $\delta = \pi/8$. This value of r_c will be reduced with a corresponding reduction in relative approach δ. As the indenter tip was moved closer to the edge, the shift in the center of contact from the indenter tip was revealed. For smaller wedge angles, placing the indenter tip directly over the wedge apex caused contact to occur over an area not extending to the edge resulting in a "pulling away" effect. Also, the two-dimensional stress singularity predictions for non-rigid indentation of a quarter space and indentation of wedges of greater than 90° were confirmed by the numerical results when contact extended to the edge. A three-dimensional study provides the additional benefit of allowing the total displacements to be considered instead of only the displacement gradients in two-dimensional analysis. The three-dimensional results displayed the loss in stiffness (reduced contact force for constant indentation depth) as the edge was approached.

As a final comment it is important to note that edges in contact display the singular or non-singular behavior shown in the paper by Dundurs and Lee (1972). If the contact is not frictionless, the order of the singularity will in general be different. The extension of the Dundurs and Lee paper to include friction has been conducted by Gdoutos and Theocaris (1975) and Comninou (1976).

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